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On the Klein-Gordon equation and hyperbolic pseudoanalytic function theory

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Abstract

Elliptic pseudoanalytic function theory was considered independently by Bers and Vekua decades ago. In this paper we develop a hyperbolic analogue of pseudoanalytic function theory using the algebra of hyperbolic numbers. We consider the Klein-Gordon equation with a potential. With the aid of one particular solution we factorize the Klein-Gordon operator in terms of two Vekua-type operators. We show that real parts of the solutions of one of these Vekua-type operators are solutions of the considered Klein-Gordon equation. Using hyperbolic pseudoanalytic function theory, we then obtain explicit construction of infinite systems of solutions of the Klein-Gordon equation with potential. Finally, we give some examples of application of the proposed procedure.

Keywords: Pseudoanalytic Function Theory, Klein-Gordon Equation, Hyperbolic Numbers.

1 Introduction

Pseudoanalytic function theory became an important part of classical complex analysis after publication of the monographs [1, 2] where the main basic results had been already established. In the recent works [3, 4] some new features and applications of this theory were discovered, first of all an intimate relation between pseudoanalytic functions and solutions of the stationary Schrödinger equation as well as a possibility to obtain explicitly complete systems of solutions for an ample class of second-order elliptic partial differential equations.

In the present work we develop a hyperbolic analogue of pseudoanalytic function theory which proves to be extremely useful for studying hyperbolic partial differential equations. We show that solutions of the Klein-Gordon equation with an arbitrary potential are closely related to certain hyperbolic pseudoanalytic functions, the result of a factorization of the Klein-Gordon operator with the aid of two Vekua-type operators. As one of the corollaries we obtain a method for explicit construction of infinite systems of solutions of the considered Klein-Gordon equation. Our approach is based on the application of the algebra of hyperbolic numbers [5, 6] instead of that of complex numbers and generalizes some earlier works dedicated to hyperbolic analytic function theory [7, 8, 9].

It should be mentioned that the elliptic and hyperbolic pseudoanalytic function theories naturally result to be quite different. Nevertheless as we show in the present work there are many important common features.

2 Hyperbolic numbers and analytic functions

It has been proven (see, e.g., [10]) that there exist essentially three possible ways to generalize real numbers into real algebras of dimension two. Indeed, each possible system can be reduced to one of the following

1. numbers $a + b\mathrm{i}$ with $\mathrm{i}^2 = -1$ (complex numbers);
2. numbers $a + b\mathrm{j}$ with $\mathrm{j}^2 = 1$ (hyperbolic numbers);
3. numbers $a + b\mathrm{k}$ with $\mathrm{k}^2 = 0$ (dual numbers).

In this article the set of hyperbolic numbers, also called duplex numbers (see, e.g., [5, 6]), will be denoted by

$$\mathbb{D} := \left\{ x + t\mathrm{j} \ : \ \mathrm{j}^2 = 1, \ x, t \in \mathbb{R} \right\} \cong \mathrm{Cl}_{\mathbb{R}}(0, 1). \tag{2.1}$$

It is easy to see that this algebra of hyperbolic numbers is commutative and contains zero divisors.

As in the case of complex numbers, we denote the real and "imaginary" parts of $z = x + t\mathrm{j} \in \mathbb{D}$ by $x = \mathrm{Re}(z)$ and $t = \mathrm{Im}(z)$. Now, by defining the conjugate as $\bar{z} := x - t\mathrm{j}$ and the hyperbolic modulus as $|z|^2 := z\bar{z} = x^2 - t^2$, we

can verify that the inverse of z whenever exists is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2}. \tag{2.2}$$

From this, we find that the set $\mathcal{NC}$ of zero divisors of $\mathbb{D}$, called the *null-cone*, is given by

$$\mathcal{NC} = \big\{ x + t\mathrm{j} \ : \ |x| = |t| \big\}.$$

It is also possible to define differentiability of a function at a point of $\mathbb{D}$ [11, 12]:

Definition 1. *Let U be an open set of $\mathbb{D}$ and $z_0 \in U$. Then, $f : U \subseteq \mathbb{D} \longrightarrow \mathbb{D}$ is said to be $\mathbb{D}$-differentiable at z_0 with derivative equal to $f'(z_0) \in \mathbb{D}$ if*

$$\lim_{\substack{z \to z_0 \\ (z - z_0 \ inv.)}} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0). \tag{2.3}$$

Here z tends to z_0 following the invertible trajectories. We also say that the function f is $\mathbb{D}$-holomorphic on an open set U if and only if f is $\mathbb{D}$-differentiable at each point of U.

Any hyperbolic number can be seen as an element of $\mathbb{R}^2$, so a function $f(x + t\mathrm{j}) = f_1(x,t) + f_2(x,t)\mathrm{j}$ can be seen as a mapping $f(x,t) = \big(f_1(x,t), f_2(x,t)\big)$ of $\mathbb{R}^2$.

Theorem 1. *Let U be an open set and $f : U \subseteq \mathbb{D} \longrightarrow \mathbb{D}$ such that $f \in C^1(U)$. Let also $f(x + t\mathrm{j}) = f_1(x,t) + f_2(x,t)\mathrm{j}$. Then f is $\mathbb{D}$-holomorphic on U if and only if*

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial t} \qquad \textit{and} \qquad \frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial t}. \tag{2.4}$$

Moreover $f' = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial x}\mathrm{j}$ and $f'(z)$ is invertible if and only if $\det \mathcal{J}_f(z) \neq 0$, where $\mathcal{J}_f(z)$ is the Jacobian matrix of f at z.

It is important to keep in mind that every hyperbolic number $x + t\mathrm{j}$ has the following unique idempotent representation

$$x + t\mathrm{j} = (x + t)\mathrm{e}_1 + (x - t)\mathrm{e}_2, \tag{2.5}$$

where $\mathrm{e}_1 = \dfrac{1 + \mathrm{j}}{2}$ and $\mathrm{e}_2 = \dfrac{1 - \mathrm{j}}{2}$. This representation is very useful because with its aid addition, multiplication and division can be done term-by-term.

The notion of holomorphicity can also be seen with this kind of notation. For this we need to define the projections $P_1, P_2 : \mathbb{D} \longrightarrow \mathbb{R}$ as $P_1(z) = x + t$ and $P_2(z) = x - t$, where $z = x + t\mathrm{j}$ as well as the following definition.

Definition 2. *We say that $X \subseteq \mathbb{D}$ is a $\mathbb{D}$-cartesian set determined by X_1 and X_2 if*

$$X = X_1 \times_e X_2 := \left\{x + t\mathrm{j} \in \mathbb{D} : x + t\mathrm{j} = w_1\mathrm{e}_1 + w_2\mathrm{e}_2, (w_1, w_2) \in X_1 \times X_2\right\}. \quad (2.6)$$

It is easy to show that if X_1 and X_2 are open domains of $\mathbb{R}$ then $X_1 \times_e X_2$ is also an open domain of $\mathbb{D}$. Now, it is possible to formulate the following theorem.

Theorem 2. *If $f_{e_1} : X_1 \longrightarrow \mathbb{R}$ and $f_{e_2} : X_2 \longrightarrow \mathbb{R}$ are real differentiable functions on the open domains X_1 and X_2 respectively, then the function $f : X_1 \times_e X_2 \longrightarrow \mathbb{D}$ defined as*

$$f(x + t\mathrm{j}) = f_{e_1}(x + t)\mathrm{e}_1 + f_{e_2}(x - t)\mathrm{e}_2, \ \forall \ x + t\mathrm{j} \in X_1 \times_e X_2 \qquad (2.7)$$

is $\mathbb{D}$-holomorphic on the domain $X_1 \times_e X_2$ and

$$f'(x + t\mathrm{j}) = f'_{e_1}(x + t)\mathrm{e}_1 + f'_{e_2}(x - t)\mathrm{e}_2, \ \forall \ x + t\mathrm{j} \in X_1 \times_e X_2. \qquad (2.8)$$

3 Hyperbolic pseudoanalytic functions

3.1 Elementary hyperbolic derivative

We will consider the variable $z = x + t\mathrm{j}$, where x and t are real variables and the corresponding formal differential operators

$$\partial_z = \frac{1}{2}\left(\partial_x + \mathrm{j}\partial_t\right) \text{ and } \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x - \mathrm{j}\partial_t\right). \qquad (3.1)$$

Notation $f_{\bar{z}}$ or f_z means the application of $\partial_{\bar{z}}$ or ∂_z respectively to a hyperbolic function $f(z) = u(z) + v(z)\mathrm{j}$. These hyperbolic operators act on sums, products, etc. just as an ordinary derivative and we have the following result in the hyperbolic function theory. We note that

$$f_z = \frac{1}{2}\Big((u_x + v_t) + (v_x + u_t)\mathrm{j}\Big) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}\Big((u_x - v_t) + (v_x - u_t)\mathrm{j}\Big).$$

In view of these operators,

$$f_z(z) = 0 \quad \Leftrightarrow \quad (u_x + v_t) + (v_x + u_t)\mathrm{j} = 0 \qquad (3.2)$$

i.e. $u_x = -v_t, \ v_x = -u_t$ and

$$f_{\bar{z}}(z) = 0 \quad \Leftrightarrow \quad (u_x + v_t) + (v_x + u_t)\mathrm{j} = 0 \qquad (3.3)$$

i.e. $u_x = v_t, \ v_x = u_t$.

Lemma 1. *Let* $f(x+t\mathrm{j}) = u(x,t) + v(x,t)\mathrm{j}$ *be a hyperbolic function where* u_x, u_t, v_x *and* v_t *exist, and are continuous in a neighborhood of* z_0. *The derivative*

$$f'(z_0) = \lim_{\substack{z\to z_0 \\ (z-z_0\ inv.)}} \frac{f(z)-f(z_0)}{z-z_0} \tag{3.4}$$

exists, if and only if

$$f_{\bar{z}}(z_0) = 0. \tag{3.5}$$

Moreover, $f'(z_0) = f_z(z_0)$ *and* $f'(z_0)$ *is invertible if and only if* $\det \mathcal{J}_f(z_0) \neq 0$.

3.2 Hyperbolic pseudoanalytic function theory

Let $z = x + t\mathrm{j}$ where $x, t \in \mathbb{R}$. The theory is based on assigning the part played by 1 and j to two essentially arbitrary hyperbolic functions F and G. We assume that these functions are defined and twice continuously differentiable in some open domain $\Omega \subset \mathbb{D}$. We require that

$$\mathrm{Im}\{\overline{F(z)}G(z)\} \neq 0. \tag{3.6}$$

Under this condition, (F,G) will be called a generating pair in Ω. Notice that $\mathrm{Im}\{\overline{F(z)}G(z)\} = \begin{vmatrix} \mathrm{Re}\{F(z)\} & \mathrm{Re}\{G(z)\} \\ \mathrm{Im}\{F(z)\} & \mathrm{Im}\{G(z)\} \end{vmatrix}$. It follows, from Cramer's theorem, that for every z_0 in Ω we can find unique constants $\lambda_0, \mu_0 \in \mathbb{R}$ such that $w(z_0) = \lambda_0 F(z_0) + \mu_0 G(z_0)$. More generally we have the following result.

Theorem 3. *Let* (F,G) *be generating pair in some open domain* Ω. *If* $w(z) : \Omega \subset \mathbb{D} \to \mathbb{D}$, *then there exist* ***unique*** *functions* $\phi(z), \psi(z) : \Omega \subset \mathbb{D} \to \mathbb{R}$ *such that*

$$w(z) = \phi(z)F(z) + \psi(z)G(z),\ \forall z \in \Omega. \tag{3.7}$$

Moreover, we have the following explicit formulas for ϕ *and* ψ:

$$\phi(z) = \frac{\mathrm{Im}[\overline{w(z)}G(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]},\ \psi(z) = -\frac{\mathrm{Im}[\overline{w(z)}F(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]}. \tag{3.8}$$

Proof. Let (F,G) be generating pair in some open domain Ω. Let $z_0 \in \Omega$ with $w(z_0) = x_1 + t_1\mathrm{j}$, $F(z_0) = x_2 + t_2\mathrm{j}$ and $G(z_0) = x_3 + t_3\mathrm{j}$. In this case, $w(z_0) = \phi(z_0)F(z_0) + \psi(z_0)G(z_0)$ with $\phi(z_0), \psi(z_0) \in \mathbb{R}$ if and only if $x_1 = \phi(z_0)x_2 + \psi(z_0)x_3$ and $t_1 = \phi(z_0)t_2 + \psi(z_0)t_3$. That is we obtain the system $AX = B$ where $A = \left(\begin{smallmatrix} x_2 & x_3 \\ t_2 & t_3 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} x_1 \\ t_1 \end{smallmatrix}\right)$ and $X = \left(\begin{smallmatrix} \phi(z_0) \\ \psi(z_0) \end{smallmatrix}\right)$ and the unique solution is $X = A^{-1}B$ where $A^{-1} = \frac{1}{\det A}\left(\begin{smallmatrix} t_3 & -x_3 \\ -t_2 & x_2 \end{smallmatrix}\right)$. Hence,

$$\begin{aligned} X &= \frac{1}{\mathrm{Im}[\overline{F(z_0)}G(z_0)]}\begin{pmatrix} t_3 & -x_3 \\ -t_2 & x_2 \end{pmatrix}\begin{pmatrix} x_1 \\ t_1 \end{pmatrix} \\ &= \frac{1}{\mathrm{Im}[\overline{F(z_0)}G(z_0)]}\begin{pmatrix} \mathrm{Im}[\overline{w(z_0)}G(z_0)] \\ -\mathrm{Im}[\overline{w(z_0)}F(z_0)] \end{pmatrix}. \end{aligned} \tag{3.9}$$

Then

$$\phi(z) = \frac{\mathrm{Im}[\overline{w(z)}G(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]}, \ \psi(z) = -\frac{\mathrm{Im}[\overline{w(z)}F(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]}, \ \forall z \in \Omega. \ \square \tag{3.10}$$

Consequently, every hyperbolic function w defined in some subdomain of Ω admits the unique representation $w = \phi F + \psi G$ where the functions ϕ and ψ are real valued. Thus, the pair (F, G) generalizes the pair $(1, \mathrm{j})$ which corresponds to hyperbolic analytic function theory. Sometimes it is convenient to associate with the function w the function $\omega = \phi + \mathrm{j}\psi$. The correspondence between w and ω is one-to-one.

We say that $w : \Omega \subset \mathbb{D} \to \mathbb{D}$ possesses at z_0 the (F, G)-derivative $\dot{w}(z_0)$ if the (finite) limit

$$\dot{w}(z_0) = \lim_{\substack{z \to z_0 \\ (z-z_0 \ inv.)}} \frac{w(z) - \lambda_0 F(z) - \mu_0 G(z)}{z - z_0} \tag{3.11}$$

exists.

The following expressions are called the characteristic coefficients of the pair (F, G):

$$\begin{aligned} a_{(F,G)} &= -\frac{\bar{F}G_{\bar{z}} - F_{\bar{z}}\bar{G}}{F\overline{G} - \overline{F}G}, \quad & b_{(F,G)} &= \frac{FG_{\bar{z}} - F_{\bar{z}}G}{F\overline{G} - \overline{F}G} \\ A_{(F,G)} &= -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \quad & B_{(F,G)} &= \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G}. \end{aligned} \tag{3.12}$$

Set (for a fixed z_0)

$$W(z) = w(z) - \lambda_0 F(z) - \mu_0 G(z), \tag{3.13}$$

the constants $\lambda_0, \mu_0 \in \mathbb{R}$ being uniquely determined by the condition

$$W(z_0) = 0. \tag{3.14}$$

Hence $W(z)$ has continuous partial derivatives if and only if $w(z)$ has. Moreover, $\dot{w}(z_0)$ exists if and only if $W'(z_0)$ does, and if it does exist, then $\dot{w}(z_0) = W'(z_0)$. Therefore, by the Lemma 1, if we suppose $w \in C^1(\Omega)$, the equation

$$W_{\bar{z}}(z_0) = 0 \tag{3.15}$$

is necessary and sufficient for the existence of (3.11). Now,

$$W(z) = \frac{\begin{vmatrix} w(z) & w(z_0) & \overline{w(z_0)} \\ F(z) & F(z_0) & \overline{F(z_0)} \\ G(z) & G(z_0) & \overline{G(z_0)} \end{vmatrix}}{\begin{vmatrix} F(z_0) & \overline{F(z_0)} \\ G(z_0) & \overline{G(z_0)} \end{vmatrix}} \tag{3.16}$$

so that (3.15) may be written in the form

$$\begin{vmatrix} w_{\bar{z}}(z_0) & w(z_0) & \overline{w(z_0)} \\ F_{\bar{z}}(z_0) & F(z_0) & \overline{F(z_0)} \\ G_{\bar{z}}(z_0) & G(z_0) & \overline{G(z_0)} \end{vmatrix} = 0 \tag{3.17}$$

and if (3.11) exists, then

$$\dot{w}(z_0) = \frac{\begin{vmatrix} w_z(z_0) & w(z_0) & \overline{w(z_0)} \\ F_z(z_0) & F(z_0) & \overline{F(z_0)} \\ G_z(z_0) & G(z_0) & \overline{G(z_0)} \end{vmatrix}}{\begin{vmatrix} F(z_0) & \overline{F(z_0)} \\ G(z_0) & \overline{G(z_0)} \end{vmatrix}}. \tag{3.18}$$

Equations (3.18) and (3.17) can be rewritten in the form

$$\dot{w} = w_z - A_{(F,G)}w - B_{(F,G)}\overline{w} \tag{3.19}$$

$$w_{\bar{z}} = a_{(F,G)}w + b_{(F,G)}\overline{w}. \tag{3.20}$$

Thus we have proved the following result.

Theorem 4. *Let (F,G) be a generating pair in some open domain Ω. Every hyperbolic function $w \in C^1(\Omega)$ admits the unique representation $w = \phi F + \psi G$ where $\phi, \psi : \Omega \subset \mathbb{D} \to \mathbb{R}$. Moreover, the (F,G)-derivative $\dot{w} = \frac{d_{(F,G)}w}{dz}$ of $w(z)$ exists and has the form*

$$\dot{w} = \phi_z F + \psi_z G = w_z - A_{(F,G)}w - B_{(F,G)}\overline{w} \tag{3.21}$$

if and only if

$$w_{\bar{z}} = a_{(F,G)}w + b_{(F,G)}\overline{w}. \tag{3.22}$$

The equation (3.22) can be rewritten in the following form

$$\phi_{\bar{z}}F + \psi_{\bar{z}}G = 0. \tag{3.23}$$

Equation (3.22) is called "hyperbolic Vekua equation" and any continuously differentiable solutions of this equation are called "hyperbolic (F,G)-pseudoanalytic functions".

If w is hyperbolic (F,G)-pseudoanalytic, the associated function $\omega = \phi + \psi \mathrm{j}$ is called hyperbolic (F,G)-pseudoanalytic of second kind.

Remark 1. *The functions F and G are hyperbolic (F,G)-pseudoanalytic, and $\dot{F} \equiv \dot{G} \equiv 0$.*

Definition 3. *Let* (F, G) *and* (F_1, G_1) *- be two generating pairs in* Ω. (F_1, G_1) *is called successor of* (F, G) *and* (F, G) *is called predecessor of* (F_1, G_1) *if*

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad \text{and} \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

The importance of this definition becomes obvious from the following statement.

Theorem 5. *Let* w *be a hyperbolic* (F, G)*-pseudoanalytic function and let* (F_1, G_1) *be a successor of* (F, G)*. If* $\dot{w} = W \in C^1(\Omega)$ *then* W *is a hyperbolic* (F_1, G_1)*-pseudoanalytic function.*

Proof. The proof in the hyperbolic case is identical to the elliptic case that we find in the book of Bers [1].

Definition 4. *Let* (F, G) *be a generating pair. Its adjoint generating pair* $(F, G)^* = (F^*, G^*)$ *is defined by the formulas*

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}. \tag{3.24}$$

The (F, G)-integral is defined as follows

$$\int_\Gamma w \, \mathrm{d}_{(F,G)} z = F(z_1) \,\mathrm{Re} \int_\Gamma G^* w \, \mathrm{d}z + G(z_1) \,\mathrm{Re} \int_\Gamma F^* w \, \mathrm{d}z \tag{3.25}$$

where Γ is a rectifiable curve leading from z_0 to z_1.

If $w = \phi F + \psi G$ is a hyperbolic (F, G)-pseudoanalytic function where ϕ and ψ are real valued functions then

$$\int_{z_0}^{z} \dot{w} \, \mathrm{d}_{(F,G)} \zeta = w(z) - \phi(z_0) F(z) - \psi(z_0) G(z). \tag{3.26}$$

This integral is path-independent and represents the (F, G)-antiderivative of $\dot{w}$. The expression $\phi(z_0)F(z) + \psi(z_0)G(z)$ in (3.26) can be seen as a "*pseudoanalytic constant*" of the generating pair (F, G) in Ω.

A continuous function $W(z)$ defined in a domain Ω will be called (F, G)-integrable if for every closed curve Γ situated in a simply connected subdomain of Ω the following equality holds

$$\oint_\Gamma W \mathrm{d}_{(F,G)} z = 0. \tag{3.27}$$

Theorem 6. *Let* W *be a hyperbolic* (F, G)*-pseudoanalytic function. Then* W *is* (F, G)*-integrable.*

Proof. It will suffice to show that if Ω is a regular domain and Γ lies within the domain of definition of W, then

$$\int_\Gamma W \mathrm{d}_{(F,G)} z \tag{3.28}$$

is zero.

From the definitions (3.12) and (3.24) we find

$$
\begin{aligned}
a_{(F^*,G^*)} &= -a_{(F,G)}, \quad A_{(F^*,G^*)} = -A_{(F,G)}, \\
b_{(F^*,G^*)} &= -\overline{B_{(F,G)}}, \quad B_{(F^*,G^*)} = -\overline{b_{(F,G)}}.
\end{aligned}
\tag{3.29}
$$

Hence we obtain

$$F^*_{\bar{z}} = -aF^* - \overline{B}\,\overline{F^*}, \quad G^*_{\bar{z}} = -aG^* - \overline{B}\,\overline{G^*}, \tag{3.30}$$

and by hypothesis

$$W_{\bar{z}} = aW - B\overline{W}, \tag{3.31}$$

where $a,\ b,\ A$ and B are the characteristics coefficients of (F,G).

Let us now use the definition (3.25) to evaluate (3.28). By using the hyperbolic Green's theorem (see [9]), we obtain

$$
\begin{aligned}
\int_\Gamma G^* W \mathrm{d}z &= 2\mathrm{j} \int\int_\Omega \partial_{\bar{z}}(G^* W)\mathrm{d}x\mathrm{d}t \\
&= 2\mathrm{j} \int\int_\Omega \Big(-aG^*W - \overline{B}\,\overline{G^*}W + G^*aW - G^*B\overline{W}\Big)\mathrm{d}x\mathrm{d}t \\
&= -4\mathrm{j} \int\int_\Omega \mathrm{Re}\Big(G^*B\overline{W}\Big)\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

which is a purely imaginary number. The same argument shows that $\int_\Gamma F^* W \mathrm{d}z$ is a pure imaginary number. Hence by definition (3.25) we find that (3.28) is zero . □

3.3 Generating sequences

Definition 5. *A sequence of generating pairs $\{(F_m, G_m)\}$ with $m \in \mathbb{Z}$, is called a generating sequence if (F_{m+1}, G_{m+1}) is a successor of (F_m, G_m). If $(F_0, G_0) = (F, G)$, we say that (F, G) is embedded in $\{(F_m, G_m)\}$.*

Definition 6. *A generating sequence $\{(F_m, G_m)\}$ is said to have period $\mu > 0$ if $(F_{m+\mu}, G_{m+\mu})$ is equivalent to (F_m, G_m) that is their characteristic coefficients coincide.*

Let w be a hyperbolic (F,G)-pseudoanalytic function. Using a generating sequence in which (F,G) is embedded we can define the higher derivatives of w by the recursion formula

$$w^{[0]} = w; \qquad w^{[m+1]} = \frac{\mathrm{d}_{(F_m,G_m)} w^{[m]}}{\mathrm{d}z}, \quad m = 1, 2, \ldots$$

Definition 7. *The formal power $Z_m^{(0)}(a, z_0; z)$ with center at $z_0 \in \Omega$, coefficient a and exponent 0 is defined as the linear combination of the generators F_m, G_m with real constant coefficients λ, μ chosen so that $\lambda F_m(z_0) + \mu G_m(z_0) = a$. The formal powers with exponents $n = 1, 2, \dots$ are defined by the recursion formula*

$$Z_m^{(n)}(a, z_0; z) = n \int_{z_0}^{z} Z_{m+1}^{(n-1)}(a, z_0; \zeta) \mathrm{d}_{(F_m,G_m)}\zeta. \tag{3.32}$$

This definition implies the following properties.

1. $Z_m^{(n)}(a, z_0; z)$ is a (F_m, G_m)- hyperbolic pseudoanalytic function of z.
2. If a' and a'' are real constants, then $Z_m^{(n)}(a' + \mathrm{j}a'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(\mathrm{j}, z_0; z)$.
3. The formal powers satisfy the differential relations

$$\frac{\mathrm{d}_{(F_m,G_m)} Z_m^{(n)}(a, z_0; z)}{dz} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$

4. The asymptotic formulas

$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0$$

hold.

4 Relationship between hyperbolic pseudoanalytic functions and solutions of the Klein-Gordon equation

4.1 Factorization of the Klein-Gordon equation

Consider the $(1+1)$-dimensional Klein-Gordon equation

$$\Big(\Box - \nu(x,t)\Big)\varphi(x,t) = 0 \tag{4.1}$$

in some domain $\Omega \subset \mathbb{R}^2$, where $\Box := \dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial t^2}$, ν and φ are real valued functions. We assume that φ is a twice continuously differentiable function.

As for the stationary two-dimensional Schrödinger equation [3] it is possible to factorize the Klein-Gordon equation with potential. By C we denote the the hyperbolic conjugation operator.

Theorem 7. *Let f be a positive particular solution of (4.1) in Ω. Then for any real valued function $\varphi \in C^2(\Omega)$ the following equalities hold*

$$\begin{aligned}(\Box - \nu)\varphi &= 4\Big(\partial_{\bar z} + \frac{f_z}{f}C\Big)\Big(\partial_z - \frac{f_z}{f}C\Big)\varphi \\ &= 4\Big(\partial_z + \frac{f_{\bar z}}{f}C\Big)\Big(\partial_{\bar z} - \frac{f_{\bar z}}{f}C\Big)\varphi.\end{aligned} \tag{4.2}$$

Proof. Consider

$$\begin{aligned}\left(\partial_{\bar{z}}+\frac{f_z}{f}C\right)\left(\partial_z-\frac{f_z}{f}C\right)\varphi &= \partial_{\bar{z}}\partial_z\varphi-\frac{|f_z|^2}{f^2}\varphi-\partial_{\bar{z}}\left(\frac{f_z}{f}\right)\varphi \\ &= \frac{1}{4}(\Box\,\varphi-\frac{\Box f}{f}\varphi)=\frac{1}{4}(\Box-\nu)\varphi.\end{aligned} \tag{4.3}$$

Thus we have the first equality in (4.2). Now application of C to both sides of (4.3) gives us the second equality in (4.2). $\Box$

Note that the operator $\partial_z-\frac{f_z}{f}I$, where I is the identity operator, can be represented in the form

$$P=\partial_z-\frac{f_z}{f}I=f\partial_z f^{-1}I. \tag{4.4}$$

Let us introduce the notation $P:=f\partial_z f^{-1}I$. From Theorem 7, if f is a positive solution of (4.1), the operator P transforms real valued solutions of (4.1) into solutions of the following hyperbolic Vekua equation

$$\left(\partial_{\bar{z}}+\frac{f_z}{f}C\right)w=0. \tag{4.5}$$

The operator ∂_z applied to a real valued function φ can be regarded as a kind of gradient. If we have $\partial_z\varphi=\Phi$ in a convex hyperbolic domain, where $\Phi=\Phi_1+\mathrm{j}\Phi_2$ is a given hyperbolic valued function such that its real part Φ_1 and imaginary part Φ_2 satisfy

$$\partial_t\Phi_1-\partial_x\Phi_2=0, \tag{4.6}$$

then we can construct φ up to an arbitrary real constant c. Indeed, we have

$$\varphi(x,t)=2\left(\int_{x_0}^{x}\Phi_1(\eta,t)\mathrm{d}\eta+\int_{t_0}^{t}\Phi_2(x_0,\xi)\mathrm{d}\xi\right)+c \tag{4.7}$$

where (x_0,t_0) is an arbitrary fixed point in the domain of interest. We will denote the integral operator in (4.7) by A:

$$A[\Phi](x,t)=2\left(\int_{x_0}^{x}\Phi_1(\eta,t)\mathrm{d}\eta+\int_{t_0}^{t}\Phi_2(x_0,\xi)\mathrm{d}\xi\right)+c \tag{4.8}$$

Thus if Φ satisfies (4.6), there exists a family of real valued functions φ such that $\partial_z\varphi=\Phi$, given by $\varphi=A[\Phi]$.

In a similar way we define the operator $\bar{A}$ corresponding to $\partial_{\bar{z}}$. The family of real-valued function φ such that $\partial_{\bar{z}}\varphi=\Phi$, where $\varphi=\overline{A}[\Phi]$, can be constructed as

$$\bar{A}[\Phi](x,t)=2\left(\int_{x_0}^{x}\Phi_1(\eta,t)\mathrm{d}\eta-\int_{t_0}^{t}\Phi_2(x_0,\xi)\mathrm{d}\xi\right)+c, \tag{4.9}$$

when

$$\partial_t \Phi_1 + \partial_x \Phi_2 = 0. \tag{4.10}$$

Note that both definitions A and $\overline{A}$ are easily extended to any simply connected domain.

Consider the operator $S = fAf^{-1}I$ applicable to any hyperbolic valued function w such that $\Phi = f^{-1}w$ satisfies condition (4.6). Then it is clear that for such w we have that $PSw = w$.

Theorem 8. *Let f be a positive particular solution of (4.1) and w be a solution of (4.5). Then the real-valued function $g = Sw$ is a solution of (4.1).*

Proof. First, let us verify that the function $\Phi = w/f$ satisfies condition (4.6). Let $w = u + \mathrm{j}v$. We find

$$\partial_t \Phi_1 - \partial_x \Phi_2 = f^{-1} \cdot \left((u_t - v_x) - \left(\frac{f_t}{f} u - \frac{f_x}{f} v \right) \right). \tag{4.11}$$

The equation (4.5) is equivalent to the system

$$u_x - v_t = -\frac{f_x}{f} u + \frac{f_t}{f} v, \quad u_t - v_x = \frac{f_t}{f} u - \frac{f_x}{f} v. \tag{4.12}$$

and we find that the expression (4.11) is zero. Hence, the function $\Phi = w/f$ satisfies (4.6) and we have $PSw = w$.

Let $Q = (\partial_{\bar{z}} + \frac{f_{\bar{z}}}{f} C)$ such that $QP = \frac{1}{4}(\Box - \nu)$ from Theorem 7. We obtain

$$PSw = w \quad \Rightarrow \quad QPSw = Qw = 0 \quad \Rightarrow \quad \frac{1}{4}(\Box - \nu)Sw = 0. \ \Box \tag{4.13}$$

4.2 The main hyperbolic Vekua equation

The Vekua equation (4.5) is closely related to another Vekua equation given by

$$\left(\partial_{\bar{z}} - \frac{f_{\bar{z}}}{f} C \right) W = 0. \tag{4.14}$$

Indeed, one can observe that the pair of functions

$$F = f \quad \text{and} \quad G = \frac{\mathrm{j}}{f} \tag{4.15}$$

is a generating pair for (4.14). The associated characteristic coefficients are then given by

$$A_{(F,G)} = 0, \quad B_{(F,G)} = \frac{f_z}{f}, \quad a_{(F,G)} = 0, \quad b_{(F,G)} = \frac{f_{\bar{z}}}{f} \tag{4.16}$$

and the (F,G)-derivative according to (3.21) is defined as follows

$$\dot{W} = W_z - \frac{f_z}{f}\overline{W} = \left(\partial_z - \frac{f_z}{f}C\right)W. \tag{4.17}$$

From Definition 3 and Theorem 5, if we compare $B_{(F,G)}$ with the coefficient in (4.5) we obtain the following statement.

Theorem 9. *If $W \in C^1(\Omega)$ is a solution of (4.14), then its (F,G)-derivative $\dot{W} = w$ is a solution of (4.5) on Ω.*

Let us now consider the (F,G)-antiderivative. Taking into account that $F^* = \mathrm{j}f$ and $G^* = 1/f$, we find

$$\begin{aligned} \int_{z_0}^{z} w(\zeta)\mathrm{d}_{(F,G)}\zeta &= f(z)\mathrm{Re}\int_{z_0}^{z}\frac{w(\zeta)}{f(\zeta)}\mathrm{d}\zeta - \frac{\mathrm{j}}{f(z)}\mathrm{Re}\int_{z_0}^{z}\mathrm{j}f(\zeta)w(\zeta)\mathrm{d}\zeta \\ &= f(z)\mathrm{Re}\int_{z_0}^{z}\frac{w(\zeta)}{f(\zeta)}\mathrm{d}\zeta + \frac{\mathrm{j}}{f(z)}\mathrm{Im}\int_{z_0}^{z}f(\zeta)w(\zeta)\mathrm{d}\zeta \end{aligned} \tag{4.18}$$

and we obtain the following statement.

Theorem 10. *If w is a solution of (4.5), then the function*

$$W(z) = \int_{z_0}^{z} w(\zeta)\mathrm{d}_{(F,G)}\zeta \tag{4.19}$$

is a solution of (4.14).

Lemma 2. *Let b be a hyperbolic function such that b_z is a real-valued function, and let $W = u + \mathrm{j}v$ be a solution of the equation*

$$W_{\bar{z}} = b\overline{W}. \tag{4.20}$$

Then u is a solution of the equation

$$\frac{1}{4}\Box u = (b\bar{b} + b_z)u \tag{4.21}$$

and v is a solution of the equation

$$\frac{1}{4}\Box v = (b\bar{b} - b_z)v. \tag{4.22}$$

Proof. We observe that under the conjugation the equation $W_{\bar{z}} = b\overline{W}$ is equivalent to $\partial_z(u - \mathrm{j}v) = \bar{b}(u + \mathrm{j}v)$. Then we obtain

$$\begin{aligned} \frac{1}{4}\Box(u + \mathrm{j}v) &= \partial_z\partial_{\bar{z}}(u + \mathrm{j}v) \\ &= b_z(u - \mathrm{j}v) + b\partial_z(u - \mathrm{j}v) \\ &= b_z(u - \mathrm{j}v) + b\bar{b}(u + \mathrm{j}v) \end{aligned} \tag{4.23}$$

and by considering the real and imaginary parts of this expression we complete the proof. □

Theorem 11. *Let W be a solution of (4.14). Then $u = \mathrm{Re}\ W$ is a solution of (4.1) and $v = \mathrm{Im}\ W$ is a solution of the equation*

$$\Big(\Box - \eta\Big)v = 0, \qquad \textit{where} \qquad \eta = -\nu + 8\frac{|f_z|^2}{f^2}. \tag{4.24}$$

Proof. Let us first show that for $b = \dfrac{f_{\bar{z}}}{f}$ then b_z is a real-valued function:

$$b_z = \frac{(\partial_z f_{\bar{z}})f - f_{\bar{z}} f_z}{f^2} = \frac{1}{4}\frac{\Box f}{f} - \frac{|f_z|^2}{f^2} = \frac{1}{4}\nu - \frac{|f_z|^2}{f^2} \in \mathbb{R}. \tag{4.25}$$

We can easily calculate that $4(b\bar{b}+b_z) = \nu$ and $4(b\bar{b}-b_z) = \eta$ such that according to Lemma 2 we find $(\Box - \nu)u = 0$ and $(\Box - \eta)v = 0$. $\Box$

Remark 2. *If we consider the case $\nu = 0$ in (4.1), then we obtain the one-dimensional wave equation $\Box\varphi = 0$ with the well known general solution $\varphi = F(x+t) + G(x-t)$, where F and G are two arbitrary real-valued functions of one variable. In this case, the potential η is then given by $\eta = 8\dfrac{F'G'}{(F+G)^2}$.*

Theorem 12. *Let u be a solution of (4.1). Then the function $v \in \ker(\Box - \eta)$, such that $W = u + \mathrm{j}v$ is a solution of (4.14), is constructed according to the formula*

$$v = -f^{-1}\overline{A}\big[\mathrm{j}f^2\partial_{\bar{z}}(f^{-1}u)\big]. \tag{4.26}$$

It is unique up to an additive term cf^{-1} where c is an arbitrary real constant.

Let v be a solution of (4.24). Then the function $u \in \ker(\Box - \nu)$, such that $W = u + \mathrm{j}v$ is a solution of (4.14), can be constructed as

$$u = -f\overline{A}\big[\mathrm{j}f^{-2}\partial_{\bar{z}}(fv)\big], \tag{4.27}$$

up to an additive term cf.

Proof. Consider $W = \phi f + \mathrm{j}\psi/f$ to be a solution of the Vekua equation (4.14). Then this equation can be rewritten in the form

$$\begin{aligned} \psi_{\bar{z}} &= -\mathrm{j}f^2\phi_{\bar{z}} \\ &= \frac{f^2}{2}(\phi_t - \mathrm{j}\phi_x). \end{aligned} \tag{4.28}$$

Taking into account that $\phi = u/f$, $(\Box - \nu)u = 0$ and $(\Box - \nu)f = 0$, we can verify that

$$\partial_t\left(\frac{f^2}{2}\phi_t\right) + \partial_x\left(\frac{f^2}{2}\phi_x\right) = 0, \tag{4.29}$$

such that we can use (4.9) and ψ is given by $\psi = -\overline{A}\big[\mathrm{j}f^2\phi_{\bar{z}}\big]$. Now, since $v = \mathrm{Im}\ W = \psi/f$ we find $v = -f^{-1}\overline{A}\big[\mathrm{j}f^2\partial_{\bar{z}}(f^{-1}u)\big]$. The function v is a solution of (4.24) due to Theorem 11. Note that as the operator $\overline{A}$ reconstructs the scalar function up to an arbitrary real constant, the function v in formula (4.26) is uniquely determined up to an additive term cf^{-1} where c is an arbitrary real constant.

The equation (4.27) is proved in a similar way. $\Box$

Example 1. *Let us illustrate the last theorem by a simple example. Considering* $f(x,t) = xt = \frac{1}{4}\big((x+t)^2 - (x-t)^2\big)$ *and* $u(x,t) = 1$ *to be two particular solutions of the wave equation in the subdomain* $0 < x < t < \infty$*, then* $v = -f^{-1}\overline{A}\big[\mathrm{j}f^2\partial_{\bar{z}}(f^{-1}u)\big] \in \ker\big(\Box - \eta\big)$*, where*

$$\eta(x,t) = 8\frac{|f_z|^2}{f^2} = 2\frac{t^2 - x^2}{x^2t^2}. \tag{4.30}$$

Explicitly, the solution v *is given by*

$$v(x,t) = \frac{x^2+t^2}{2xt}. \tag{4.31}$$

4.3 Generating sequence of the main Vekua equation

The first step in the construction of a generating sequence for the main Vekua equation (4.14) is the construction of a generating pair for the equation (4.5) which, as was shown previously, is a successor of the main Vekua equation. For this, one of the possibilities consists in constructing another pair of solutions of (4.14). Then their (F,G)-derivatives will give us solutions of (4.5).

Consider the main Vekua equation (4.14) which is equivalent to the equation

$$\phi_{\bar{z}}F + \psi_{\bar{z}}G = 0, \tag{4.32}$$

where $W = \phi F + \psi G$, $F = f$ and $G = \mathrm{j}/f$. The equation (4.32) can be rewritten explicitly as the following system of partial differential equations

$$\begin{aligned} \phi_x f^2 - \psi_t &= 0, \\ \psi_x - \phi_t f^2 &= 0. \end{aligned} \tag{4.33}$$

Let us suppose that f and ϕ are functions of some real variable $\rho = \rho(x,t)$, i.e. $f = f(\rho)$ and $\phi = \phi(\rho)$. The system (4.33) then becomes

$$\begin{aligned} \psi_x &= \phi'\rho_t f^2, \\ \psi_t &= \phi'\rho_x f^2. \end{aligned} \tag{4.34}$$

The compatibility condition for this system implies

$$\partial_x\Big(\phi'\rho_x f^2\Big) - \partial_t\Big(\phi'\rho_t f^2\Big) = 0, \tag{4.35}$$

which is equivalent to the equation

$$\phi'' + \left(\frac{\Box\rho}{4|\rho_z|^2} + 2\frac{f'}{f}\right)\phi' = 0, \tag{4.36}$$

for $|\rho_z|^2 \neq 0$. We assume now that $\frac{\Box\rho}{4|\rho_z|^2}$ is a function of ρ, i.e.

$$s(\rho) = \frac{\Box\rho}{4|\rho_z|^2}. \tag{4.37}$$

Hence, under this hypothesis, we can integrate (4.36) and obtain

$$\phi'(\rho) = \frac{\mathrm{e}^{-S(\rho)}}{f^2}, \tag{4.38}$$

where $S(\rho) = \displaystyle\int_{\rho_0}^{\rho} s(\sigma)\,\mathrm{d}\sigma$.

We can now integrate (4.38) and (4.34) to obtain a solution $W = \phi F + \psi G$ of (4.14). However, since we are interested to find a solution of (4.5), i.e. the (F, G)-derivative $\dot{W}$, we need ϕ_z and ψ_z which are given explicitly by

$$\begin{aligned} \phi_z &= \frac{\mathrm{e}^{-S}\rho_z}{f^2}, \\ \psi_z &= \frac{\mathrm{j}}{2}\,\mathrm{e}^{-S}\rho_z. \end{aligned} \tag{4.39}$$

Thus, a solution $w_1 = \phi_z F + \psi_z G$ of (4.5) is given by

$$w_1 = \frac{3}{2}\,\mathrm{e}^{-S}\frac{\rho_z}{f}. \tag{4.40}$$

In much the same way we can construct another solution of (4.5) looking for $\psi = \psi(\rho)$. The system (4.33) then becomes

$$\begin{aligned} \phi_x &= \frac{\psi'\rho_t}{f^2}, \\ \phi_t &= \frac{\psi'\rho_x}{f^2}. \end{aligned} \tag{4.41}$$

and $\psi'(\rho) = f^2\mathrm{e}^{-S(\rho)}$. Calculating ϕ_z and ψ_z we find

$$\begin{aligned} \phi_z &= \frac{\mathrm{j}}{2}\,\mathrm{e}^{-S}\rho_z, \\ \psi_z &= f^2\mathrm{e}^{-S}\rho_z, \end{aligned} \tag{4.42}$$

which give us another solution w_2 of (4.5):

$$w_2 = \frac{3}{2}\mathrm{j}\,\mathrm{e}^{-S}\rho_z f. \tag{4.43}$$

Hence, for the function $\Phi = \mathrm{j}\,\mathrm{e}^{-S}\rho_z \neq 0$ we have found a generating pair for the Vekua equation (4.5) given by (eliminating the constant $\frac{3}{2}$ in w_1 and w_2):

$$\begin{aligned} \big(F_1, G_1\big) &= \left(\mathrm{j}\,\mathrm{e}^{-S}\rho_z f,\ \mathrm{j}\,\mathrm{e}^{-S}\rho_z\frac{\mathrm{j}}{f}\right) \\ &= \big(\Phi F, \Phi G\big). \end{aligned} \tag{4.44}$$

Indeed, we have

$$\operatorname{Im}\big(\overline{F_1}G_1\big) = \operatorname{Im}\big(|\Phi|^2\overline{F}G\big) = -\mathrm{e}^{-S}|\rho_z|^2 \neq 0. \tag{4.45}$$

The following step is to construct the generating pair (F_2, G_2). For this we should find two other solutions of (4.5), equivalent to $\phi_{\bar{z}}F_1 + \psi_{\bar{z}}G_1 = 0$. Then to obtain (F_2, G_2) we calculate the (F_1, G_1)-derivative of these solutions. Using the same assumptions and the same method as in the previous case, we obtain

$$\big(F_2, G_2\big) = \big(\Phi^2 F, \Phi^2 G\big). \tag{4.46}$$

The generalization of results (4.44) and (4.46) is given in the next theorem which allows us to obtain a generating sequence wherein the generating pair (F, G) of (4.14) is embedded. Let us note that under the assumption (4.37) the function Φ is a "hyperbolic analytic function", i.e. $\Phi_{\bar{z}} = 0$. Indeed, we have

$$\begin{aligned}
\Phi_{\bar{z}} &= \mathrm{j}\Big((\partial_{\bar{z}}\mathrm{e}^{-S})\rho_z + \tfrac{1}{4}\mathrm{e}^{-S}\Box\rho\Big) \\
&= -\frac{1}{4}\mathrm{j}\,\mathrm{e}^{-S}\Big(4s|\rho_z|^2 - \Box\rho\Big) = 0.
\end{aligned}$$

Theorem 13. *Let f be a nonvanishing solution of (4.1) such that $f = f(\rho)$, $\rho = \rho(x, t)$, and $\frac{\Box\rho}{4|\rho_z|^2}$ is a function of ρ denoted by $s(\rho)$. Let also the function Φ such that $\Phi = \mathrm{j}\,\mathrm{e}^{-S(\rho)}\rho_z \neq 0$, where $S(\rho) = \int_{\rho_0}^{\rho} s(\sigma)\mathrm{d}\sigma$. Then the generating pair (F, G) with $F = f$ and $G = \mathrm{j}/f$ is embedded in the generating sequence (F_m, G_m) where $F_m = \Phi^m F$, $G_m = \Phi^m G$ and $m \in \mathbb{Z}$.*

Proof. First, let us show that (F_m, G_m) is a generating pair in $\mathbb{Z}$. Indeed, we find

$$\operatorname{Im}\big(\overline{F_m}G_m\big) = \operatorname{Im}\big(|\Phi|^{2m}\overline{F}G\big) = (-1)^m\mathrm{e}^{-2mS}|\rho_z|^{2m} \neq 0.$$

To complete the proof, we need to show that $\big\{(F_m, G_m)\big\}$ forms a generating sequence, i.e. (F_m, G_m) is a successor of (F_{m-1}, G_{m-1}):

$$a_{(F_m,G_m)} = a_{(F_{m-1},G_{m-1})} \quad \text{and} \quad b_{(F_m,G_m)} = -B_{(F_{m-1},G_{m-1})}. \tag{4.47}$$

The coefficients $a_{(F_m,G_m)}$, $b_{(F_m,G_m)}$ and $B_{(F_m,G_m)}$ can be calculated in terms of $a_{(F,G)}$, $b_{(F,G)}$ and $B_{(F,G)}$, respectively, by taking into account that $\Phi_{\bar{z}} = 0$. We obtain

$$\begin{aligned}
&a_{(F_m,G_m)} = |\Phi|^{2m}a_{(F,G)} = 0, \qquad b_{(F_m,G_m)} = \left(\frac{\Phi}{\overline{\Phi}}\right)^m b_{(F,G)}, \\
&B_{(F_m,G_m)} = \left(\frac{\Phi}{\overline{\Phi}}\right)^m B_{(F,G)}.
\end{aligned} \tag{4.48}$$

Therefore, the first equality in (4.47) is verified. Taking into account (4.16) and (4.48), the second equality in (4.47) is reduced to

$$\overline{\Phi} f_z + \Phi f_{\bar{z}} = 0 \quad \Leftrightarrow \quad f'(\overline{\Phi}\rho_z + \Phi\rho_z) = 0. \tag{4.49}$$

Since $\Phi = \mathrm{j}\, \mathrm{e}^{-S(\rho)}\rho_z$ it is easy to observe that (4.49) is valid. □

This last theorem allow us to calculate the generating sequence (F_m, G_m) for a large class of potentials $\nu(x,t)$ in the Klein-Gordon equation (4.1). The importance of this result appears in the following theorem.

Theorem 14. *Let f be a particular solutions of (4.1) and let (F, G) be the generating pair in some open domain Ω with $F = f$ and $G = \mathrm{j}/f$. Then*

$$\mathrm{Re}\ Z^{(n)}(a, z_0; z), \qquad n = 0, 1, 2, \ldots$$

are solutions of the Klein-Gordon equation (4.1).

Proof. From property 1 of the definition 7 we see that $Z^{(n)}(a, z_0; z)$ is a hyperbolic (F, G)-pseudoanalytic function. Hence $Z^{(n)}(a, z_0; z)$ satisfies (4.14) and the real parts are solutions of (4.1) from Theorem 11. □

Example 2. *As an example of this theorem, we consider the Klein-Gordon equation (4.1) with the potential $\nu(x,t) = t^2 - x^2$ in the "time-like" subdomain $0 < x < t < \infty$. A particular solution of this equation is given by $f(\rho) = \mathrm{e}^{\rho^2}$, where we have defined $\rho(x,t) = \sqrt{xt}$. In this case the function $\frac{\Box\rho}{4|\rho_z|^2}$ is a function of ρ given by $s(\rho) = -1/\rho$, with $S(\rho) = -\ln\rho$ and $\Phi = \frac{z}{4} \neq 0$. Let us construct the first formal powers $Z^{(n)}(1, 4\mathrm{j}; z)$ and $Z^{(n)}(\mathrm{j}, 4\mathrm{j}; z)$. By the definition 7 we have*

$$\begin{aligned} Z^{(0)}(1,4\mathrm{j};4\mathrm{j}) &= 1, & Z^{(0)}(\mathrm{j},4\mathrm{j};4\mathrm{j}) &= \mathrm{j}, \\ &= \lambda_1 F(4\mathrm{j}) + \mu_1 G(4\mathrm{j}), & &= \lambda_2 F(4\mathrm{j}) + \mu_2 G(4\mathrm{j}), \\ &= \lambda_1 + \mathrm{j}\mu_1, & &= \lambda_2 + \mathrm{j}\mu_2, \end{aligned}$$

such that $\lambda_1 = \mu_2 = 1$ and $\mu_1 = \lambda_2 = 0$. Hence, we obtain

$$\begin{aligned} Z^{(0)}(1,4\mathrm{j};z) &= \lambda_1 F(z) + \mu_1 G(z), & Z^{(0)}(\mathrm{j},4\mathrm{j};z) &= \lambda_2 F(z) + \mu_2 G(z) \\ &= \mathrm{e}^{xt}, & &= \mathrm{j}\mathrm{e}^{-xt}. \end{aligned}$$

Now, from the formula (3.32), if we want to construct $Z^{(1)}(1, 4\mathrm{j}; z)$ and $Z^{(1)}(\mathrm{j}, 4\mathrm{j}; z)$ we need to calculate first $Z_1^{(0)}(1, 4\mathrm{j}; z)$ and $Z_1^{(0)}(\mathrm{j}, 4\mathrm{j}; z)$. First note that the generating pair (F_1, G_1) is given by

$$F_1 = \frac{1}{4} z\mathrm{e}^{xt} \qquad \textit{and} \qquad G_1 = \frac{\mathrm{j}}{4} z\mathrm{e}^{-xt}.$$

Hence, from definition 7 we obtain

$$\begin{aligned} Z_1^{(0)}(1,4\mathrm{j};4\mathrm{j}) &= 1, & Z_1^{(0)}(\mathrm{j},4\mathrm{j};4\mathrm{j}) &= \mathrm{j}, \\ &= \lambda_3 F_1(4\mathrm{j}) + \mu_3 G_1(4\mathrm{j}), & &= \lambda_4 F_1(4\mathrm{j}) + \mu_4 G_1(4\mathrm{j}), \\ &= \lambda_3 \mathrm{j} + \mu_3, & &= \lambda_4 \mathrm{j} + \mu_4, \end{aligned}$$

which implies that $\mu_3 = \lambda_4 = 1$ *and* $\lambda_3 = \mu_4 = 0$ *and*

$$\begin{aligned} Z_1^{(0)}(1, 4\mathrm{j}; z) &= \lambda_3 F_1(z) + \mu_3 G_1(z), & Z_1^{(0)}(\mathrm{j}, 4\mathrm{j}; z) &= \lambda_4 F_1(z) + \mu_4 G_1(z) \\ &= \frac{\mathrm{j}}{4} z \mathrm{e}^{-xt}, & &= \frac{1}{4} z \mathrm{e}^{xt}. \end{aligned}$$

¿From definition (3.32), we obtain

$$Z^{(1)}(a, 4\mathrm{j}; z) = \int_0^z Z_1^{(0)}(a, 4\mathrm{j}; \zeta) \mathrm{d}_{(F,G)}\zeta$$

and (3.24) gives $F^* = \mathrm{j}f$ *and* $G^* = 1/f$. *Now using (3.25) we find*

$$\begin{aligned} Z^{(1)}(1, 4\mathrm{j}; z) &= \frac{1}{4}\left(\mathrm{e}^{xt}\mathrm{Re}\int_0^z \mathrm{j}\mathrm{e}^{-2x't'}\zeta d\zeta + \mathrm{j}\mathrm{e}^{-xt}\mathrm{Re}\int_0^z \zeta \mathrm{d}\zeta\right), \\ Z^{(1)}(\mathrm{j}, 4\mathrm{j}; z) &= \frac{1}{4}\left(\mathrm{e}^{xt}\mathrm{Re}\int_0^z \zeta d\zeta + \mathrm{j}\mathrm{e}^{-xt}\mathrm{Re}\int_0^z \mathrm{j}\mathrm{e}^{2x't'}\zeta \mathrm{d}\zeta\right), \end{aligned}$$

where $\zeta = x' + \mathrm{j}t'$. *Evaluating these integrals, we obtain*

$$\mathrm{Re}\int_0^z \zeta \mathrm{d}\zeta = \mathrm{Re}\int_0^1 \epsilon(x + \mathrm{j}t)(x + \mathrm{j}t)\mathrm{d}\epsilon = \frac{x^2 + t^2}{2}$$

and

$$\mathrm{Re}\int_0^z \mathrm{j}\mathrm{e}^{\pm 2x't'}\zeta \mathrm{d}\zeta = \mathrm{Re}\int_0^1 \mathrm{j}(x + \mathrm{j}t)^2 \epsilon \mathrm{e}^{\pm 2\epsilon^2 xt}\mathrm{d}\epsilon = \mathrm{e}^{\pm xt}\sinh(xt),$$

such that

$$\begin{aligned} Z^{(1)}(1, 4\mathrm{j}; z) &= \frac{1}{4}\left(\sinh(xt) + \mathrm{j}\frac{x^2 + t^2}{2}\mathrm{e}^{-xt}\right), \\ Z^{(1)}(\mathrm{j}, 4\mathrm{j}; z) &= \frac{1}{4}\left(\frac{x^2 + t^2}{2}\mathrm{e}^{xt} + \mathrm{j}\sinh(xt)\right). \end{aligned}$$

Now, from the formula (3.32), if we want to find $Z^{(2)}(1, 4\mathrm{j}; z)$ *and* $Z^{(2)}(\mathrm{j}, 4\mathrm{j}; z)$ *we need to calculate first* $Z_1^{(1)}(1, 4\mathrm{j}; z)$ *and* $Z_1^{(1)}(\mathrm{j}, 4\mathrm{j}; z)$*; those are themselves obtained from* $Z_2^{(0)}(1, 4\mathrm{j}; z)$ *and* $Z_2^{(0)}(\mathrm{j}, 4\mathrm{j}; z)$.

The generating pair (F_2, G_2) *is given by*

$$F_2 = \left(\frac{z}{4}\right)^2 \mathrm{e}^{xt} \quad \textit{and} \quad G_2 = \mathrm{j}\left(\frac{z}{4}\right)^2 \mathrm{e}^{-xt},$$

which allows us to calculate $Z_2^{(0)}(1, 4\mathrm{j}; z)$ *and* $Z_2^{(0)}(\mathrm{j}, 4\mathrm{j}; z)$. *We find*

$$Z_2^{(0)}(1, 4\mathrm{j}; z) = \left(\frac{z}{4}\right)^2 \mathrm{e}^{xt}, \quad Z_2^{(0)}(\mathrm{j}, 4\mathrm{j}; z) = \mathrm{j}\left(\frac{z}{4}\right)^2 \mathrm{e}^{-xt}.$$

To obtain $Z_1^{(1)}(1, 4\mathrm{j}; z)$ *and* $Z_1^{(1)}(\mathrm{j}, 4\mathrm{j}; z)$, *we need the adjoint generating pair of* (F_1, G_1), *i.e.*

$$F_1^* = 4\mathrm{j}\frac{\mathrm{e}^{xt}}{z}, \qquad G_1^* = 4\frac{\mathrm{e}^{-xt}}{z}.$$

Using (3.32), we find

$$\begin{aligned} Z_1^{(1)}(1,4\mathrm{j};z) &= \tfrac{1}{16}(x+\mathrm{j}t)\left(\tfrac{x^2+t^2}{2}\mathrm{e}^{xt} + \mathrm{j}\sinh(xt)\right) \\ Z_1^{(1)}(\mathrm{j},4\mathrm{j};z) &= \tfrac{1}{16}(x+\mathrm{j}t)\left(\sinh(xt) + \mathrm{j}\tfrac{x^2+t^2}{2}\mathrm{e}^{-xt}\right). \end{aligned}$$

Finally, by considering again (3.32), we obtain

$$\begin{aligned} Z^{(2)}(1,4\mathrm{j};z) &= \tfrac{\mathrm{e}^{xt}}{64}\Big[(x^2+t^2)^2 + 4xt + 2\big(\cosh(2xt) - \sinh(2xt) - 1\big)\Big] \\ &\quad + \tfrac{\mathrm{j}}{64}\tfrac{x^2+t^2}{xt}\Big[\mathrm{e}^{xt}\big(4xt\sinh(2xt) - 1\big) \\ &\quad + 2\mathrm{e}^{-xt}\cosh(xt)\big(\cosh(xt) + \sinh(xt)\big) - 1\Big], \\ Z^{(2)}(\mathrm{j},4\mathrm{j};z) &= -\tfrac{1}{64}\tfrac{x^2+t^2}{xt}\Big[\mathrm{e}^{xt}\big(\sinh(2xt) - \cosh(2xt)\big) - 4xt\sinh(xt) + \mathrm{e}^{-xt}\Big] \\ &\quad - \tfrac{\mathrm{j}}{64}\mathrm{e}^{-xt}\Big[(x^2+t^2)^2 + 4\cosh(xt)\big(\sinh(xt) + \cosh(xt)\big) - 4\big(xt+1\big)\Big]. \end{aligned}$$

Using Theorem 14, we can now verify that the real parts of $Z^{(n)}(1,4\mathrm{j};z)$ and $Z^{(n)}(\mathrm{j},4\mathrm{j};z)$ with $n = 0,1,2,\dots$ are solutions of the Klein-Gordon equation with potential $\nu(x,t) = t^2 - x^2$.

Example 3. *We are now considering the Klein-Gordon equation (4.1) with potential*

$$\nu(x,t) = \frac{1}{4}\left(\frac{1}{(t+1)^2} - \frac{1}{(x+1)^2}\right) \tag{4.50}$$

on the time-like subdomain $0 < x < t < \infty$. A particular solution of this equation is given by $f(x,t) = \sqrt{(x+1)(t+1)}$. We denote $\rho = (x+1)(t+1)$. In this case it is easy to see that the function $\frac{\Box\rho}{4|\rho_z|^2}$ is zero, therefore a function of ρ. We obtain $\Phi = \frac{z}{2} + \mathrm{e}_1$, where e_1 is the idempotent constant of (2.5). Let us calculate the first formal powers $Z^{(n)}(1,t_0\mathrm{j};z)$ and $Z^{(n)}(\mathrm{j},t_0\mathrm{j};z)$, where $t_0 > 0$. By definition 7 we find

$$Z^{(0)}(1,t_0\mathrm{j};z) = \alpha^{-1}\sqrt{(x+1)(t+1)}, \quad Z^{(0)}(\mathrm{j},t_0\mathrm{j};z) = \frac{\mathrm{j}\alpha}{\sqrt{(x+1)(t+1)}}.$$

where $\alpha = \sqrt{t_0+1}$. From (3.32), in order to construct $Z^{(1)}(1,t_0\mathrm{j};z)$ and $Z^{(1)}(\mathrm{j},t_0\mathrm{j};z)$ we first need $Z_1^{(0)}(1,t_0\mathrm{j};z)$ and $Z_1^{(0)}(\mathrm{j},t_0\mathrm{j};z)$. These functions are calculating from the generating pair (F_1,G_1) given by

$$F_1(z) = \left(\frac{z}{2} + \mathrm{e}_1\right)\sqrt{(x+1)(t+1)}, \quad G_1(z) = \left(\frac{\mathrm{j}z}{2} + \mathrm{e}_1\right)\frac{1}{\sqrt{(x+1)(t+1)}}.$$

Using this generating pair and by definition 7 we obtain

$$Z_1^{(0)}(1,t_0\mathrm{j};z) = -\frac{z+2\mathrm{e}_1}{\alpha t_0(t_0+2)}\sqrt{(x+1)(t+1)}+\mathrm{j}\frac{\alpha(t_0+1)(z+2\mathrm{e}_1)}{t_0(t_0+2)}\frac{1}{\sqrt{(x+1)(t+1)}}$$

$$Z_1^{(0)}(\mathrm{j},t_0\mathrm{j};z) = \frac{(t_0+1)(z+2\mathrm{e}_1)}{\alpha t_0(t_0+2)}\sqrt{(x+1)(t+1)}-\mathrm{j}\frac{\alpha(z+2\mathrm{e}_1)}{t_0(t_0+2)}\frac{1}{\sqrt{(x+1)(t+1)}}.$$

We note that $F^* = -\mathrm{j}\sqrt{(x+1)(t+1)}$ *and* $G^* = 1/\sqrt{(x+1)(t+1)}$ *such that we are now able to calculate* $Z^{(1)}(1,t_0\mathrm{j};z)$ *and* $Z^{(1)}(\mathrm{j},t_0\mathrm{j};z)$*. We find*

$$Z^{(1)}(1,t_0\mathrm{j};z) = \sqrt{(x+1)(t+1)}\left[\tfrac{-1}{\alpha t_0(t_0+2)}\left(\tfrac{x^2+t^2}{2}+x+t\right)+\tfrac{\alpha(t_0+1)}{t_0(t_0+2)}\ln\left[(x+1)(t+1)\right]\right]$$
$$+\tfrac{\mathrm{j}}{\sqrt{(x+1)(t+1)}}\left[\tfrac{1}{2\alpha t_0(t_0+2)}\Big(2(x+t)(xt+1)+(x^2t^2+4xt+x^2+t^2)\Big)\right.$$
$$\left.-\tfrac{\alpha(t_0+1)}{t_0(t_0+2)}\left(\tfrac{x^2+t^2}{2}+x+t\right)\right]$$

and

$$Z^{(1)}(\mathrm{j},t_0\mathrm{j};z) = \sqrt{(x+1)(t+1)}\left[\tfrac{t_0+1}{\alpha t_0(t_0+2)}\left(\tfrac{x^2+t^2}{2}+x+t\right)-\tfrac{\alpha}{t_0(t_0+2)}\ln\left[(x+1)(t+1)\right]\right]$$
$$+\tfrac{\mathrm{j}}{\sqrt{(x+1)(t+1)}}\left[\tfrac{-(t_0+1)}{2\alpha t_0(t_0+2)}\Big(2(x+t)(xt+1)+(x^2t^2+4xt+x^2+t^2)\Big)\right.$$
$$\left.+\tfrac{\alpha}{t_0(t_0+2)}\left(\tfrac{x^2+t^2}{2}+x+t\right)\right].$$

Again here, we can verify that the real parts of $Z^{(1)}(1,t_0\mathrm{j};z)$ *and* $Z^{(1)}(\mathrm{j},t_0\mathrm{j};z)$ *are solutions of the Klein-Gordon equation with potential (4.50).*

5 Conlusions

We proved that the Klein-Gordon equation can be reduced to a hyperbolic Vekua equation of the form (4.14) and as a consequence under quite general conditions an infinite system of solutions of the Klein-Gordon equation can be constructed explicitly as a real part of the corresponding set of formal powers. Meanwhile in the elliptic theory this result gave us [4] a complete system of solutions (of a corresponding Schrödinger equation) it is an open question what part of the kernel of the Klein-Gordon operator is determined by the obtained solutions.

One of the main results of elliptic pseudoanalytic function theory is the so-called similarity principle [1, 13, 14]. It is interesting and important to find what is a corresponding fact in the hyperbolic case.

The reduction of the Klein-Gordon equation with an arbitrary potential to a Vekua-type hyperbolic first order equation gives the possibility to apply concepts and ideas from pseudoanalytic function theory to linear second-order wave equations. Besides some first applications presented in this work, questions related to initial and boundary value problems, existence and construction of special classes of solutions, large-time behaviour of solutions (closely related to a similarity principle) and others may receive a new development effort.

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